Homology in Abelian Lattice Models

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Abstract

We study abelian lattice gauge theory defined on a simplicial complex with arbitrary topology. The use of dual objects allows one to reformulate the theory in terms of different dynamical variables; however, we avoid the use of the dual cell complex entirely. Topological modes which are present in the transformation now appear as homology classes, in contrast to the cohomology modes found in the dual cell picture. Irregularities of dual cell complexes do not arise in this approach. We treat the two and three dimensional cases in detail.

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1 Introduction

Duality transformations have been studied in statistical systems since the early work of Kramers and Wannier [1]. It is a striking result that the high and low temperature properties of some theories are related by this method. There have been many other applications of this idea especially in the case of hypercubic lattices; see [2] for an extensive review. Looking at duality from the more general framework of a simplicial complex, one finds that topological modes generically enter into these duality transformations [3, 4]. Even the two dimensional Ising model on a simple square lattice (torus) has such modes.

While theories defined on hypercubic lattices are particularly simple to study since those lattices are self-dual, one may be interested in approximating continuum models defined on spaces with different topology. Unfortunately, the usual notion of the dual cell complex [5] associated to a given simplicial complex is in general very irregular in spite of the regularity of simplicial complexes; duality may map simplicial objects into polygons of various type. In [4], it was shown that dual theories, defined on dual cell complexes, generally have topological modes which are in correspondence with cohomology classes on the dual cell complex.

In this paper, we point out that duality can be considered without reference to dual cell complexes, and the cohomological modes which previously entered the analysis will now appear as homology classes on the original simplicial complex. While duality changes the nature of the dynamical variables in a given theory, we will here use only the original simplicial complex in formulating the dual theory.

In the following section we review the essentials of simplicial complexes and homology. Applications of our approach to duality will be illustrated in d-dimensional abelian gauge theory. The two dimensional case will then be fully analyzed on a general combinatorial 2-manifold and the partition function reduced to a single mode sum for any plaquette based action (not necessarily ones which are subdivision invariant). Finally, we treat the three dimensional case in some detail.

2 Simplicial Complexes and Homology

Let us begin by recalling some standard material on simplicial complexes and homology; we refer to [5] for a complete treatment.

Intuitively, a simplicial complex is a collection of simplices of various dimensions (points, line segments, triangles, etc.) which are glued together in a regular way. More formally, let $V = \{v_1, ..., v_{N_0}\}$ be a collection of N_0 elements which we will call the *vertex set.* A *simplicial complex* K is a collection of finite nonempty subsets of V such that if $\sigma \in K$ so is every nonempty subset of σ. An element of K is called a *simplex* and its *dimension* is one less than the number of vertices it contains. We picture the 1-dimensional simplex $\{v_i, v_j\}$ as the line segment connecting two distinct points in Euclidean space. Similarly, $\{v_i, v_j, v_k\}$ can be pictured as the triangle with the three indicated vertices. An *orientation* of a simplex $\{v_0, ..., v_m\}$, denoted $[v_0, ..., v_m]$, is an equivalence class of the ordering of the vertices according to even and odd permutations. This gives direction to a line segment and to the circulation around the boundary of a triangle, and so on for higher dimensional simplices.

Let $K^{(m)} = \{\sigma_\alpha\}$ denote the collection of all oriented m-simplices in K. The group of $m-$*chains* on K with coefficients in an abelian group G, denoted by $C_m(K, G)$, is defined to be the set of all finite linear combinations $\sum_\alpha n_\alpha \, \sigma_\alpha$, $n_\alpha \in G$, with the natural componentwise addition of chains as the group operation. A *boundary* operator $\partial_m : C_m(K, G) \to C_{m-1}(K, G)$ is defined on a given simplex

$$\partial_m[v_0, ..., v_m] = \sum_{i=0}^{m} (-1)^i \, [v_0, ..., \bar{v}_i, ..., v_m] \quad , \tag{1}$$

(where we have omitted the vertex corresponding to $\bar{v}_i$) and then extended to all of $C_m(K, G)$ by linearity. Let $Z_m(K, G)$ denote the kernel of ∂_m and $B_m(K, G)$ the image of ∂_{m+1}; the *homology group* $H_m(K, G)$ is then defined as the quotient group Z_m/B_m. $Z_m(K, G)$ is called the group of m-cycles and $B_m(K, G)$ the group of m-boundaries. It is a nontrivial theorem [5] that the homology groups are topological invariants, and hence independent of subdivision of the complex. We will have no need of either cohomology or dual block (cell) complexes in this paper.

Applications to physical systems naturally focus on discrete approximations to smooth manifolds. One fact we will use is that a simplicial complex K which models a smooth manifold M of dimension d without boundary, consists of d-simplices which are glued pairwise along faces of dimension $d-1$, so all $(d-1)$-simplices in K are common to precisely two d-simplices. For example, we can represent the three dimensional sphere S^3 as the boundary of a single 4-simplex,

$$\partial\,[0,1,2,3,4] = [1,2,3,4] - [0,2,3,4] + [0,1,3,4] - [0,1,2,4] + [0,1,2,3] \ . \quad (2)$$

In this case the complex K consists of the above 3-simplices together with all their subsimplices. Note, for example, that the 2-simplex $[2,3,4]$ is common to only the first two 3-simplices in the list, in accordance with the pairwise gluing condition.

3 Duality in Gauge Theory

We begin our analysis with the case of Z_P lattice gauge theory on a closed d dimensional simplicial complex K. By definition, such a theory is specified by an action S which is a function of the link variables U_{ij} only through the holonomy

$$U_{[i,j,k]} = U_{ij} U_{jk} U_{ki} \ . \quad (3)$$

The partition function of Z_P lattice gauge theory is defined as a sum over all link variables $U_{ij} \in Z_P$, which we represent multiplicatively, and a Boltzmann weight factor for every 2-simplex in the simplicial complex K [6],

$$\mathcal{Z}_d = P^{-N_1} \sum_{\{U_{ij}\}} \prod_{\Delta \in K^{(2)}} \exp[S(U_\Delta)] \ . \quad (4)$$

The factor of P^{-N_1}, where N_1 is the number of 1-simplices in K, serves only to normalize the group volumes to unity.

The character expansion of the Boltzmann weight is the finite sum

$$\exp[S(U)] = \sum_{n=0}^{P-1} b_n \, U^n \ , \quad (5)$$

where the P coefficients $\{b_n\}$ can be considered to be the parameters of the theory. Indeed, one can invert this relation to obtain,

$$b_n = \frac{1}{P} \sum_{U \in Z_P} U^{-n} \exp[S(U)] \ . \tag{6}$$

It is usually required that the Boltzmann weight be insensitive to the orientations of the holonomies, $\exp[S(U)] = \exp[S(U^{-1})]$, and this translates into the condition $b_{P-n} = b_n$. Without this restriction, one must specify which orientations are being used in (4). Our analysis does not require this assumption, but we will see that some formula simplify if it holds.

Let us introduce an integer $n_\Delta \in \{0, ..., P-1\}$ for each 2-simplex Δ, so $\mathcal{Z}_d$ becomes,

$$P^{-N_1} \sum_{\{U_{ij}\}} \prod_{\Delta \in K^{(2)}} \sum_{\{n_\Delta\}} b_{n_\Delta} \, U_\Delta^{n_\Delta} \ . \tag{7}$$

The collection of the n_Δ for all 2-simplices in K may be viewed as a 2-chain, which we can represent explicitly as,

$$n = \sum_{\Delta \in K^{(2)}} n_\Delta \, \Delta \ . \tag{8}$$

Now rearrange the order of factors, the idea being to collect all terms proportional to each link variable U_{ij}; we have

$$\mathcal{Z}_d = P^{-N_1} \sum_{\{n_\Delta\}} \prod_{\Delta \in K^{(2)}} b_{n_\Delta} \prod_{[i,j] \in K^{(1)}} (\sum_{U_{ij}} (\prod_{\Delta \supset [i,j]} U_{ij}^{\varepsilon([i,j],\Delta)\, n_\Delta})) \ , \tag{9}$$

where the last product in this equation is over all 2-simplices which contain the specified link $[i,j]$. The factor $\varepsilon([i,j],\Delta)$ of ± 1 explicitly records whether $[i,j]$ occurs in Δ with positive or negative orientation. Using the representation of a mod-P delta function,

$$\sum_{U \in Z_P} U^n = P \, \delta(n) \ , \tag{10}$$

one obtains,

$$\mathcal{Z}_d = \sum_{\{n_\Delta\}} \prod_{\Delta \in K^{(2)}} b_{n_\Delta} \prod_{[i,j] \in K^{(1)}} \delta(\sum_{\Delta \supset [i,j]} \varepsilon([i,j],\Delta)\, n_\Delta) \ . \tag{11}$$

Notice that the sum in the delta function is over all 2-simplices which contain $[i, j]$. Moreover, it is precisely the condition that n be a 2-cycle ($\partial n = 0$). The partition function is then simply,

$$\mathcal{Z}_d = \sum_{n \in Z_2(K, Z_P)} \prod_{\Delta \in K^{(2)}} b_{n_\Delta} \quad . \tag{12}$$

Now, we can decompose the 2-cycles in the following way,

$$Z_2(K, Z_P) = H_2(K, Z_P) \oplus \partial_3 C_3(K, Z_P) \quad , \tag{13}$$

so the first part of the sum represents those 2-cycles which are nontrivial, and the second those which are trivial in the sense that they are boundaries of 3-chains. In this "dual" picture, we see that the new dynamical variables are the 3-chains, together with a finite number of topological modes in correspondence with $H_2(K, Z_P)$. However, in summing over all of $C_3(K, Z_P)$, we would generically overcount the number of independent 2-cycles n since ∂_3 may have a kernel. To simplify the computation of this kernel, we restrict P to be a prime number so that Z_P is an algebraic field and all the chain and cycle groups are just vector spaces over Z_P.

By definition of the boundary map we have

$$\partial_m : C_m(K, Z_P) \to C_{m-1}(K, Z_P) \quad . \tag{14}$$

Taken together with the definition of homology, the two relations

$$\begin{aligned} dim\ Im(\partial_m) &= dim\ B_{m-1} = dim\ C_m - dim\ Z_m \quad , \\ dim\ H_m &= dim\ Z_m - dim\ B_m \end{aligned} \tag{15}$$

hold, and we can solve the recursion formula

$$dim\ Z_m = dim\ C_m + dim\ H_{m-1} - dim\ Z_{m-1} \tag{16}$$

which has the boundary condition $dim\ Z_0 = dim\ C_0 = N_0$. For the situation at hand, we have that

$$dim\ ker(\partial_3) = N_3 - N_2 + N_1 - N_0 + h_2 - h_1 + h_0 \quad , \tag{17}$$

where $N_m = dim\ C_m$ is the number of m-simplices in K, and $h_m = dim\ H_m(K, Z_P)$. For a connected complex K, which we always assume, $h_0 = 1$. Taking account of this kernel, we have

$$\mathcal{Z}_d = P^{-dim\, Z_3(K,Z_P)} \sum_{B\in H_2(K,Z_P)} \sum_{C\in C_3(K,Z_P)} \prod_{\Delta\in K^{(2)}} b_{(B+\partial C)_\Delta} \quad , \tag{18}$$

with $dim\ Z_3$ given explicitly by (17).

The fact that ∂_3 has a kernel means that theory as formulated in (18) has some gauge invariance; the number of gauge degrees of freedom being precisely equal to the dimension of this kernel. This amount of gauge redundancy is, however, far less than in the original link based formulation.

In the dual theory, we see that the new Boltzmann weight is just proportional to $b_{(B+\partial_3 C)}$, with the variables B and C taking their values in the additive group $Z_P = \{0, ..., P-1\}$. We can easily revert to multiplicative notation if we wish,

$$V_{[i,j,k,l]} = \exp[\frac{2\pi i}{P}\ C_{[i,j,k,l]}] \quad , \quad W_\Delta = \exp[\frac{2\pi i}{P}\ B_\Delta] \quad , \tag{19}$$

and the dual action is a function of the product of the W and V variables. The V variables that will enter a term $\hat{S}_\Delta$ in the dual action will be all those which have Δ as a face.

The analysis of $U(1)$ gauge theories is very similar to the Z_P case. If we replace the unit volume Z_P group integration measure,

$$\frac{1}{P} \sum_{U\in Z_P} \rightarrow \frac{1}{2\pi} \int_0^{2\pi} d\theta \quad , \tag{20}$$

were $U_{jk} = e^{i\theta_{jk}}$ is now the link variable associated to the link $[j,k]$, the partition function is defined to be,

$$\mathcal{Z}_d[U(1)] = (2\pi)^{-N_1} \prod_{[i,j]\in K^{(1)}} \int_0^{2\pi} d\theta_{ij} \prod_{\Delta\in K^{(2)}} \exp[S(U_\Delta)] \quad . \tag{21}$$

A character expansion for the Boltzmann weight $\exp[S(U)]$ is, in this case, nothing other than a Fourier series,

$$\sum_{n=-\infty}^{\infty} b_n\, U^n \quad . \tag{22}$$

The analysis leading up to equation (12) applies here as well, only with the integer coefficient group Z replacing Z_P, and we obtain

$$\mathcal{Z}_d[U(1)] = \sum_{n \in Z_2(K,Z)} \prod_{\Delta \in K^{(2)}} b_{n_\Delta} \quad . \tag{23}$$

To go beyond this general expression, some attention to the issue of gauge fixing is required. The decomposition in (13) can of course be used (with the coefficient group Z), but we cannot sum over redundant gauge field copies (which are in correspondence with the group of 3-cycles) since each would introduce an infinite factor via a mode sum over Z.

4 $d = 2$ Gauge theory

Gauge theory on Riemann surfaces is particularly simple, and even in the nonabelian case, the partition function for lattice gauge theory has been computed as a function of the genus [7, 8]. However, that analysis depended on choosing a Boltzmann weight which was subdivision invariant; our analysis of the abelian case will make no such restriction.

Equation (18) for the partition function of Z_P lattice gauge theory is completely general. For $d = 2$, there are no 3-chains, so $C_3 = 0$ and $dim\, Z_3 = 0$; our partition function then reduces to a sum over $H_2(K, Z_P)$,

$$\mathcal{Z}_2 = \sum_{B \in H_2(K,Z_P)} \prod_{\Delta \in K^{(2)}} b_{B_\Delta} \quad . \tag{24}$$

For an oriented manifold, $dim\ H_2(K, Z_P) = 1$, and we have just one mode sum to perform. The generator of H_2 is easily deduced from the definition of K, and B is then just a multiple of that generator. In the four vertex complex of the 2-sphere, for example, the generator of H_2 is given by

$$[1, 2, 3] - [0, 2, 3] + [0, 1, 3] - [0, 1, 2] \quad . \tag{25}$$

In fact, the fundamental class of any manifold is such a sum of d-simplices with signs. In our partition function we are summing over multiples k of this generator so each b_{B_Δ} factor is then either b_k or $b_{-k} = b_{P-k}$. When the action S is independent of the orientation of the holonomy (so $b_k = b_{P-k}$), as

is usually assumed, the partition function for Z_P gauge theory (P need not be a prime number here) on an orientable surface reduces to

$$\mathcal{Z}_2 = \sum_{k=0}^{P-1} b_k^{N_2} \quad . \tag{26}$$

It is independent of the genus and only depends on the number of 2-simplices in the triangulation. As such, one can see which Boltzmann weight factors have a continuum limit by examining the $N_2 \to \infty$ limit of (26).

There is no essential difference in the analysis of the $U(1)$ case since $Z_3(K, Z) = 0$, and we have,

$$\mathcal{Z}_2[U(1)] = \sum_{k=-\infty}^{\infty} b_k^{N_2} \quad . \tag{27}$$

5 $d = 3$ Gauge Theory

In three dimensions, the situation is nontrivial but still very regular, owing to the fact that each 2-simplex in K is the face of precisely two distinct 3-simplices. This gives rise to an action which is a function of only two dynamical variables for each 2-simplex. For a closed, oriented 3-manifold, the Euler characteristic

$$N_3 - N_2 + N_1 - N_0 \tag{28}$$

vanishes and $dim\, Z_3$ given in (17) reduces to $h_2 - h_1 + 1 = h_3 = 1$. The last equality follows from the universal coefficient and duality theorems [5]. Alternatively, we can view this as a specific confirmation of our general formula (17), since we know that in three dimensions $Z_3(K, Z_P) = H_3(K, Z_P)$ (as $B_3(K, Z_P) = 0$).

Let us illustrate the formula in detail for the complex (2) of the 3-sphere. Since $H_2(S^3, Z_P) = 0$, we have only to sum over all 3-chains in computing the partition function. An arbitrary 3-chain C can be written as

$$C = c_0\,[1,2,3,4] - c_1\,[0,2,3,4] + c_2\,[0,1,3,4] - c_3\,[0,1,2,4] + c_4\,[0,1,2,3] \quad , \tag{29}$$

where each c_a is an element of $\{0, ..., P-1\}$ and arithmetic is all modulo P. We need not have included the minus signs in (29), but they make the following expressions more symmetrical. The partition function on this complex (assuming for simplicity that the action is independent of the orientation of the holonomy) then reduces to,

$$\frac{1}{P} \sum_{c_0,\dots,c_4} b_{c_3-c_4}\, b_{c_2-c_4}\, b_{c_2-c_3}\, b_{c_1-c_4}\, b_{c_1-c_3}\, b_{c_1-c_2}\, b_{c_0-c_4}\, b_{c_0-c_3}\, b_{c_0-c_2}\, b_{c_0-c_1} \ . \tag{30}$$

As we explained above, the kernel of ∂_3 is one dimensional and we have the gauge freedom to arbitrarily set any one of the c_a variables to zero. This is substantially less gauge freedom than we had in the original link based formulation.

Equation (18) has been checked numerically on a small cubic lattice with a specific action and gauge group Z_2. Since the 3-torus T^3 has $H_2(T^3, Z_2) = Z_2 \oplus Z_2 \oplus Z_2$, we have a total of $2^3 = 8$ topological modes, and each of the three generators is represented by an embedded 2-torus. For a $3 \times 3 \times 3$ lattice, the dual partition function can be summed exactly and compared to a Monte Carlo approximation to the original gauge theory formulation (this has many more modes and cannot be summed exactly). This numerical check reveals that the individual topological sectors of the dual theory are generally distinct, and some make a negative contribution to the partition function.

The $U(1)$ gauge theory on a closed, oriented 3-manifold, parallels the above, only we are forced to gauge fix the dual theory to get a meaningful result. One has,

$$\mathcal{Z}_3[U(1)] = \sum_{B \in H_2(K,Z)} \sum_{C \in C_3'(K,Z)} \prod_{\Delta \in K^{(2)}} b_{(B+\partial C)_\Delta} \ , \tag{31}$$

where $C_3'(K, Z)$ is the gauge fixed set of all 3-chains. The later differs from $C_3(K, Z)$ only in having set a single chosen component to an arbitrary value in Z.

6 Concluding Remarks

The appearance of homology modes in duality transformations is generic, and not specific to the abelian gauge theory that we treated in this paper. These modes are analogous to the cohomology modes which are present in dual cell complex formulations [4].

Given that duality exchanges strongly and weakly coupled theories, the need to gauge fix the dual of $U(1)$ lattice gauge theory is not really surprising. This parallels the continuum situation where weak coupling is treated perturbatively, and gauge fixing is required.

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